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On the impatience implications of Paretian social welfare functions

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Abstract

In this paper, we investigate the impatience implications resulting from the assumption of existence of a Paretian social welfare function (SWF) aggregating infinite utility streams. We show, for very general program spaces, that the set of utility streams, at which the SWF exhibits impatience, has the power of the continuum. In the context of a more special program space, which has figured prominently in the literature, we establish that this set is dense, so that even if there is a point in the program space at which the SWF does not exhibit impatience, there are points close to it at which it does. If the Paretian SWF is continuous (in the sup metric), we show that impatience is generic: the collection of points, at which the SWF *does not* exhibit impatience, is a closed, nowhere dense set.

JEL classification: D60, D70, D90

Keywords: Impatience condition; Pareto axiom; Social welfare function; Generic impatience

1. Introduction

This paper is concerned with the following question. If infinite utility streams are aggregated with a Paretian social welfare function (SWF),[1] will the SWF *necessarily* exhibit a preference for advancing the timing of future satisfactions? Welfare functions that reflect such a preference are said to exhibit *impatience*.

The question has a rather long history. Irving Fisher (1930) felt that it was desirable "to find the principles which fix the terms on which present and future goods exchange, without restricting

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[1] The concept of a social welfare function is formally defined in Section 2.1.

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ourselves in advance to the thesis that, always and necessarily, present goods command a premium over future goods".

The first modern treatment of the question is, of course, contained in the seminal contribution of Koopmans (1960). He suggested a set of postulates that a welfare function (aggregating infinite streams of utility or consumption bundles) ought to satisfy and showed that for any welfare function satisfying the postulates, "impatience prevails at least in certain areas of the program space" (see Koopmans, 1960, p. 288). His postulates included continuity of the welfare function (in the sup metric), stationarity and non-complementarity.[2] The analysis of Koopmans was subsequently refined by Koopmans et al. (1964), who showed that a property of *time perspective* holds for such welfare functions, from which the earlier result on impatience can be obtained and extended to a larger part of the program space.[3]

Focusing on the case of social welfare functions aggregating infinite utility streams, Diamond (1965) imposed the Pareto axiom as a fundamental postulate, and showed that if a Paretian SWF is continuous in the sup metric, then, under a non-complementarity axiom,[4] it must exhibit a form of "eventual impatience".[5]

Burness (1973) avoided imposing non-complementarity axioms in getting his "eventual impatience" result, but he assumed that the SWF is continuously differentiable.[6]

Our investigation differs from the literature in that we do not impose *any condition* on the Paretian SWF. We seek to understand whether the *existence* of a Paretian SWF itself implies that "impatience prevails at least in certain areas of the program space". If we employ a method, recently used by Basu and Mitra (2003) to address a related question, we see that this implication does indeed obtain under a very general set-up (Theorem 1). In fact, we can show that a Paretian SWF must exhibit impatience for a set of utility streams of the power of the continuum.[7]

Svensson (1980) has shown that there exists a Paretian social welfare order (SWO)[8] which is "equitable", so that it does not exhibit impatience in any part of the program space. Taken together, these results indicate that the impatience implication follows from the real-valued *representation* of Paretian social welfare orders. It is possible that Koopmans (1960) did not foresee this. He viewed his study as following the lines of Ragnar Frisch, but he used a utility function rather than a preference relation to formulate his postulates, noting that: "To limit the mathematical difficulties, the postulates of the present study are in terms of a utility function, with the understanding that an alternative with higher utility is always preferred over one with lower utility, and indifference exists between alternatives of equal utility" (Koopmans, 1960, p. 288). In fact, if we accept the Pareto axiom, it is the postulate of a utility function that creates the mathematical difficulty that equal treatment of all generations is not possible.

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2 He did not directly impose the Pareto axiom (as we do), since his exercise could also be interpreted as applying to an individual's welfare function over streams of consumption bundles and a monotonicity postulate on such welfare functions would, perhaps, not be a natural requirement on the entire domain.

3 The subsequent literature, focusing on the theory and applications of a "stationary ordinal utility", not necessarily of the additive separable form, is substantial. A comprehensive account of it can be found in Becker and Boyd (1997).

4 See Section 2.3 for the non-complementarity axiom used by Diamond.

5 See Section 3.1 for an elaboration of Diamond's result.

6 Burness (1976) also studied the impatience implications following from the assumption of *separability* (a strong form of non-complementarity) of the welfare function.

7 Burness (1973) gives an example of a Paretian SWF which does not exhibit impatience in any part of the program space. However, his program space does not fit into our framework (which includes the standard framework used by Koopmans (1960), Diamond (1965), Svensson (1980) and others).

8 A social welfare order is formally defined in Section 2.1.

A framework which has figured prominently in the literature is one where the set of utilities possible for any generation is the closed interval $Y = [0, 1]$ and social welfare orders and functions are defined on the program space $X = Y^{\mathbb{N}}$, which represents the set of infinite utility streams. If we confine our analysis to this program space, equipped with the sup metric, then the implication of the existence of any Paretian SWF is that the set of points in X, at which the SWF exhibits impatience, must be *dense* in X (Theorem 2). Thus, even if there is a point in X, at which the SWF does not exhibit impatience, there are always points close to it (in the sup metric) at which it does exhibit impatience.

Given the denseness result, it is natural to enquire whether impatience is, in fact, a *generic* feature of Paretian SWFs defined on this program space. We illustrate, by means of an example, that impatience may not be generic for a Paretian SWF (see Section 3.3.1). The SWF in our example is not continuous (in the sup metric) and this motivates an investigation of the impatience implications of *continuous* Paretian SWFs. We show (Theorem 3) that if a Paretian SWF is continuous (in the sup metric), then the set of points, at which the SWF exhibits impatience, is open, so that near any point at which the SWF exhibits impatience, there are only points at which the SWF exhibits impatience. Impatience is therefore a generic phenomenon in the sense that the set of points, at which the SWF *does not* exhibit impatience, is a closed, nowhere dense set in the program space.[9]

Continuity (in the sup metric) of a Paretian SWO implies that it can be represented by a continuous (in the sup metric) Paretian SWF (see Diamond, 1965). Thus, we have: (a) the observation that for Paretian social welfare orders, the assumption of continuity of such an order has significantly stronger implications for impatience than the assumption of representability and (b) the striking result that while a Paretian SWO has *no* implications for impatience, a *continuous* (in the sup metric) Paretian SWO has the very strong implication that impatience is in fact *generic*.

2. Preliminaries

2.1. Notation and definitions

Let $\mathbb{N}$ denote, as usual, the set of natural numbers $\{1, 2, 3, \ldots\}$ and let $\mathbb{R}$ denote the set of real numbers. Let Y be any subset of $\mathbb{R}$ with at least two distinct elements and let X denote the set $Y^{\mathbb{N}}$. Then, X is the domain of utility sequences (also, referred to as "utility streams" or "utility profiles") that we are interested in: thus, $x \equiv (x_1, x_2, \ldots) \in X$ if and only if $x_n \in Y$ for all $n \in \mathbb{N}$. Given $x \in X$ and $M, N \in \mathbb{N}$, with $M > N$, we denote by $x(M, N)$ the sequence $x' \in X$, defined by:

$$x'_M = x_N,\ x'_N = x_M \text{ and } x'_n = x_n \text{ for all } n \neq N, M$$

For $x, y \in X$, we write $x \geq y$ if $x_n \geq y_n$ for all $n \in \mathbb{N}$; and we write $x > y$ if $x \geq y$ and $x \neq y$. Given $x \in X$ and $t \in \mathbb{N}$, we denote $(x_1, \ldots, x_t)$ by ${}_1x_t$ and $(x_{t+1}, x_{t+2}, \ldots)$ by ${}_{t+1}x$.

A *social welfare ordering* is a binary relation, $\succsim$, on X, which is complete and transitive. We associate with $\succsim$ its symmetric and asymmetric components in the usual way. Thus, we write $x \sim y$ when $x \succsim y$ and $y \succsim x$ both hold; and, we write $x \succ y$ when $x \succsim y$ holds, but $y \succsim x$ does not hold. A *social welfare function* (SWF) is a function, W, from X to $\mathbb{R}$. A SWF W *represents* a

[9] Consequently, the set of points, at which the SWF does not exhibit impatience, is a set of the *first category* in the sense of Baire.

SWO $\succsim$ if for all $x, y \in X$:

$$x \succsim y \text{ iff } W(x) \geq W(y)$$

2.2. The Pareto axiom and an impatience condition

Let $\succsim$ be a SWO on X. A basic requirement on a SWO is that it satisfy the Pareto axiom.

2.2.1. Pareto axiom

If $x, y \in X$ and $x > y$ then $x \succ y$.

The Pareto axiom requires that the SWO be positively sensitive to the well-being of each generation. A SWO satisfying the axiom of Pareto will be called a *Paretian SWO*. If W is a SWF which represents a Paretian SWO, then it is called a *Paretian SWF*. Thus, if W is a Paretian SWF, and $x, y \in X$ with $x > y$, then $W(x) > W(y)$.

The impatience condition we will be concerned with can be stated as follows.

2.2.2. Impatience condition

A SWO $\succsim$ exhibits impatience at $x \in X$ if there exist $M, N \in \mathbb{N}$ with $M > N$ such that:

$$\text{Either (i) } x_M > x_N \text{ and } x(M, N) \succ x \text{ or (ii) } x_M < x_N \text{ and } x \succ x(M, N)$$

If W is a SWF, representing a SWO $\succsim$ which exhibits impatience at $x \in X$, then there exist $M, N \in \mathbb{N}$ with $M > N$ such that:

$$(x_N - x_M)(W(x(M, N)) - W(x)) < 0$$

Thus, the concept of impatience involves the comparison of the utility streams x and $x(M, N)$. Impatience at x means that there exist some dates M and N (with $M > N$) such that if a larger (smaller) "future" utility x_M is exchanged with a smaller (larger) "current" utility x_N, then the resulting utility stream, $x(M, N)$, is preferred to (worse than) x. So the expression "exhibits impatience at x" really means "exhibits impatience at x, for *some* choice of dates".

Put differently, a SWO $\succsim$ *does not* exhibit impatience at $x \in X$ if the following two properties are satisfied: (i) whenever $M, N \in \mathbb{N}$ with $M > N$ and $x_M > x_N$, $x \succsim x(M, N)$ holds and (ii) whenever $M, N \in \mathbb{N}$ with $M > N$ and $x_M < x_N$, $x(M, N) \succsim x$ holds. Note that the two properties are (trivially) satisfied if x is a constant utility stream. We say that $\succsim$ exhibits *pure patience* at $x \in X$, if x is *not* a constant utility stream and $\succsim$ does not exhibit impatience at x.[10]

Note that our notion of impatience is deliberately chosen to be a weak one. In particular, it allows for social welfare orders or functions, which exhibit "impatient behavior" for some parts of the program space, and "patient" behavior for other parts of the program space, X. This is because the objective of our study is precisely to see what impatience implications result from postulating the existence of a Paretian SWF.

Given that objective, it is nevertheless important to define the concept of impatience for social welfare orders (which might or might not be representable by social welfare functions). This is because we know (from Svensson, 1980) that there exist Paretian SWOs which treat all generations equitably, and therefore do not exhibit impatience according to any reasonable definition of the

[10] If a SWO $\succsim$ exhibits impatience at $x \in X$, we find it convenient to refer to x as an "impatient point"; similarly, if SWO $\succsim$ exhibits pure patience at $x \in X$, we refer to x as a "purely patient point". These should be considered to be short-cuts for the corresponding formal concepts, which we have defined.

term; and a principal purpose of the paper is to show that there is a significant difference if we look at Paretian SWFs instead.

2.3. Non-complementarity

The term "non-complementarity" expresses the idea that preferences over some parts of the time horizon are independent of the utilities attained at other time points. The non-complementarity axiom used in Diamond (1965) is introduced here, since it facilitates the understanding of Diamond's impatience result (compared to ours), which is discussed in Section 3.1.

2.3.1. Non-complementarity axiom (NC)

(i) For every $x, y \in X$, $u, v \in Y$ and $t \in \mathbb{N}$, $(u,_{t+1}\, x) \succsim (u,_{t+1}\, y)$ implies $(v,_{t+1}\, x) \succsim (v,_{t+1}\, y)$ and (ii) for every $x, y \in X$ and $t \in \mathbb{N}$, $({}_1x_{t,t+1}x) \succsim ({}_1y_{t,t+1}x)$ implies $({}_1x_{t,t+1}y) \succsim ({}_1y_{t,t+1}y)$.

3. Impatience implications of Paretian SWFs

3.1. Existence of impatience

This sub-section presents the main existence result of the paper, which can be stated as follows. Given a domain $X = Y^{\mathbb{N}}$ (where Y has at least two distinct elements), and *any* Paretian SWF $W : X \to \mathbb{R}$, there must exist some profile in X at which W exhibits impatience. In fact, we establish the considerably stronger result that the set of utility profiles for which W must exhibit impatience is of the power of the continuum. Since X also has the power of the continuum, the subset of points of X at which W exhibits impatience is equivalent in cardinality to the entire set X. We formally state the result as follows:

Theorem 1. *Suppose $\succsim$ is a Paretian SWO on X, which can be represented by $W : X \to \mathbb{R}$. Define*:

$$I = \{x \in X :\ W \text{ exhibits impatience at } x\}$$

Then I is of the power of the continuum.

Remarks.

(i) Burness (1973) gives an example of a social welfare function, $W : X \to \mathbb{R}$ on $X = \ell_1^+$, which is both Paretian and for which it is not true that it exhibits some impatience. His example is:

$$W(x) = 1 - \mathrm{e}^{\left[-\sum_{t=1}^{\infty} \alpha_t x_t\right]}$$

where $\{\alpha_t\}$ is a sequence of positive real numbers, which is monotonically increasing and bounded above. Such a welfare function exhibits *pure patience* at every non-constant utility stream $x \in X$. However, the utility space X used by Burness is not of the form $Y^{\mathbb{N}}$, which is the standard framework of analysis used by Koopmans (1960, 1972), Diamond (1965), Svensson (1980), Basu and Mitra (2003). By restricting the space to utility profiles for which the infinite sum of the components converge, one restricts the utility choices available to future generations. The space where $X = Y^{\mathbb{N}}$ is rich in this respect and allows all generations to have the same utility possibility set Y, regardless of how far away in the future they are.

(ii) It is important to note that our result is true on any space of the form $X = Y^{\mathbb{N}}$, where $Y \subset \mathbb{R}$ has at least two distinct elements. For example, consider the case where, at each point in time, only two states of the world are possible, i.e., $Y = \{0, 1\}$. Even with this simple structure, given Theorem 1, we know that it is impossible to define a Paretian SWF that exhibits pure patience everywhere in the program space X. This particular implication of Theorem 1 is close to the result obtained in Theorem 1 in Basu and Mitra (2003): while the implication cannot directly be derived from their result, it can be derived from their method of proof. Theorem 1, of course, goes beyond this particular implication in saying something about the set of points where the Paretian SWF exhibits impatience.

(iii) Our impatience concept is similar in spirit to Burness (1973, Definition 3, p. 498); it is, strictly speaking, non-comparable to the impatience concept used by Diamond (1965).

Diamond's concept of "eventual impatience" is the following. A social welfare order $\succsim$ exhibits *eventual impatience* if for each $x \in X$ and $\epsilon > 0$, there is $s \in \mathbb{N}$, such that for all $t \geq s$ for which $|x_1 - x_t| \geq \epsilon$, we have: (i) $x \succ x(t, 1)$ if $x_t < x_1$ and (ii) $x(t, 1) \succ x$ if $x_t > x_1$. Thus, on utility profiles that converge to the pre-determined initial generation's utility,[11] one cannot make any welfare inference using Diamond's concept.[12] Our impatience condition allows us to meaningfully talk about impatience on a larger subset of the program space.

On utility profiles where both impatience conditions are applicable, Diamond establishes a stronger form of impatience, by imposing the non-complementarity axiom NC (of Section 2.3), together with continuity of the Paretian SWF in the sup metric.[13] We refrain from making any assumption on the Paretian SWF beyond its existence.

3.2. Utility streams exhibiting impatience are dense

Theorem 1 does not provide a precise idea of the structure of the set of impatient points in the set X. In this subsection, we address this issue in the special framework in which the set $Y = [0, 1]$, and the set X is endowed with the sup metric, denoted henceforth by d.[14] We show that the set I of impatient points is *dense* in the metric space (X,d). Thus, even if there is a point at which the SWF does not exhibit impatience, there are always points "close to it" at which it does.

Theorem 2. *Suppose $\succsim$ is a Paretian SWO on $X = [0, 1]^{\mathbb{N}}$, which can be represented by $W : X \to \mathbb{R}$. Define*:

$$I = \{x \in X : \ W \text{ exhibits impatience at } x\}$$

Then, I is a dense subset of (X,d).

Theorem 2 provides a surprisingly strong result on the structure of points where the Paretian social welfare function exhibits impatience. Based solely on the existence of a Paretian SWF in the metric space (X, d), the result indicates, loosely speaking, that impatience is exhibited in all parts of the program space. It is useful to recall here that, in contrast, the existence of a Paretian SWO is compatible with impatience being exhibited in *no* part of the program space.

[11] Diamond formulates the impatience concept with the fixed initial period as period 1. However, the analysis suggests that his impatience results are not sensitive to this choice.

[12] For a discussion of this aspect of Diamond's result, see Burness (1973, p. 504).

[13] Diamond also assumes that $Y = [0, 1]$ throughout his analysis.

[14] This can be considered to be the standard framework in this literature; it appears in the analysis of Koopmans (1960, 1972), Diamond (1965), Svensson (1980).

3.3. *Genericity of impatience*

Given a Paretian SWF, W, in the metric space (X, d), we say that impatience is *generic* if the complement of the set of points of X at which W exhibits impatience, is a closed, nowhere dense subset of X. In this subsection, we explore the circumstances under which impatience is generic.

3.3.1. *Existence and structure of purely patient utility streams*

We first provide an example of a Paretian SWF, which exhibits pure patience at some points in the program space. Furthermore, for this example, the set of impatient points is not open in the sup metric topology. Consequently, impatience is, in general, *not* a generic phenomenon for Paretian SWFs.

Example 1. Let $Y = [0, 1]$ and $X = Y^{\mathbb{N}}$. Define:

$$X^1 = \left\{x \in X : \sum_{t=1}^{\infty} x_t < 1\right\}; \quad X^2 = \{x \in X : x \notin X^1\}$$

Define $W : X \to \mathbb{R}$ by:

$$W(x) = \begin{cases} \displaystyle\sum_{t=1}^{\infty} x_t, & \text{if } x \in X^1 \\ \displaystyle 1 + \sum_{t=1}^{\infty}(x_t/2^{t-1}), & \text{if } x \in X^2 \end{cases}$$

Clearly, W is a social welfare function. Define $\succsim$ by:

For all $x, y \in X$, $x \succsim y$ if and only if $W(x) \geq W(y)$

Then, $\succsim$ is a social welfare order, which is represented by W.

We check that W is Paretian. Let $x, x' \in X$, with $x' > x$. We consider two cases: (i) $x \in X^1$ and (ii) $x \in X^2$. In case (i), we consider two sub-cases: (a) $x' \in X^1$ and (b) $x' \in X^2$. In case (i)(a), we have:

$$W(x') = \sum_{t=1}^{\infty} x'_t > \sum_{t=1}^{\infty} x_t = W(x)$$

the strict inequality following from $x' > x$. In case (i)(b), we have:

$$W(x') \geq 1 > \sum_{t=1}^{\infty} x_t = W(x)$$

In case (ii), since $x \in X^2$ and $x' > x$, we must also have $x' \in X^2$. Thus,

$$W(x') = 1 + \sum_{t=1}^{\infty}(x'_t/2^{t-1}) > 1 + \sum_{t=1}^{\infty}(x_t/2^{t-1}) = W(x)$$

the strict inequality following from $x' > x$.

The SWF, W, exhibits pure patience at every non-zero x in X^1, since x is not a constant utility stream and $W(x) = W(x(M, N))$ for every $M, N \in \mathbb{N}$.

We now show that there is a point $\bar{x} \in X$ such that W exhibits impatience at $\bar{x}$, and every open ball (in the sup metric) with center $\bar{x}$ contains a purely patient point. Consider $\bar{x} \in X$, defined by:

$$\bar{x} = (1/2, 1/4, 1/8, 1/16, \ldots)$$

Then, we have:

$$\sum_{t=1}^{\infty} \bar{x}_t = 1$$

and so:

$$W(\bar{x}) = 1 + \sum_{t=1}^{\infty} (\bar{x}_t / 2^{t-1})$$

Choosing $N = 1$ and $M = 2$, we have $M > N$, and:

$$\bar{x}(M, N) = \bar{x}(2, 1) = (1/4, 1/2, 1/8, 1/16, \ldots)$$

Then, clearly,

$$\sum_{t=1}^{\infty} \bar{x}(2, 1)_t = 1$$

so that:

$$W(\bar{x}(2, 1)) = 1 + \sum_{t=1}^{\infty} (\bar{x}(2, 1)_t / 2^{t-1})$$

Then, it is easy to check that:

$$W(\bar{x}) - W(\bar{x}(M, N)) = (1/8) > 0$$

so that W exhibits impatience at $\bar{x}$ (since $\bar{x}_1 = 1/2 > 1/4 = \bar{x}_2$).

Define, for $n = 1, 2, 3, \ldots$

$$x^n = (1/2, 1/4, \ldots, 1/2^n, 0, 0, 0, \ldots)$$

Then, clearly $x^n \in X^1$ for each n, and therefore W exhibits pure patience at x^n for each n. However, we have:

$$\sup_t |x_t^n - \bar{x}_t| \to 0 \text{ as } n \to \infty \tag{1}$$

Thus, every open ball in X (with the sup metric) with center $\bar{x}$ must contain some point, x^n, at which W exhibits pure patience. That is, the set of impatient points is not open in X (with the sup metric), and impatience is not generic for the Paretian SWF, W.

The SWO $\succsim$ violates continuity in the sup metric.[15] To see this, define $x' = (0, 1, 0, 0, 0, 0, \ldots)$. Then $x' \in X_2$ and $W(x') = (3/2)$. Since $W(x^n) < 1$ for each $n \in \mathbb{N}$, we have $x' \succ x^n$ for each $n \in \mathbb{N}$. If $\succsim$ were continuous in the sup metric, then we must have $x' \succsim \bar{x}$, since (1) holds. But, clearly, $W(\bar{x}) = 1 + \sum_{t=1}^{\infty} (\bar{x}_t / 2^{t-1}) > (3/2) = W(x')$.

Remarks. The example provides an instance of a Paretian SWO $\succsim$, which has a real-valued representation, but which is not continuous in the sup metric.

[15] Consequently, the SWF, X, also violates continuity in the sup metric.

3.3.2. Impatience is generic for continuous Paretian SWFs

Example 1 prompts us to explore the impatience implications of Paretian SWFs, which are, in addition, continuous in the sup metric.

Continuing to work in the metric space (X, d), where $X = [0, 1]^{\mathbb{N}}$ and d is the sup-metric, if we are given a Paretian SWF which is continuous in the sup metric, then we can show that the set of points, at which the SWF exhibits impatience, is open in (X, d), so that near any point at which the SWF exhibits impatience, there can only be points at which the SWF exhibits impatience.

Theorem 3. *Suppose* $\succsim$ *is a Paretian SWO on* $X = [0, 1]^{\mathbb{N}}$, *which can be represented by* $W : X \to \mathbb{R}$. *Define*:

$$I = \{x \in X : W \text{ exhibits impatience at } x\}$$

If W is continuous on X in the sup metric topology, then I is an open subset in (X, d).

Remarks.

(i) Using Theorems 2 and 3, we know that the set of all impatient points is open and dense in (X, d). Hence the set, *NI*, of utility streams in *X*, at which *W does not* exhibit impatience, is closed, and, since *I* is dense in *X*, the set *NI* cannot contain any open set. Thus, *NI* is a closed, *nowhere dense* set (Exercise 31a, p. 161, Royden, 1988), and in this sense we can say that impatience is generic.[16] It follows from this that *NI* is also of the *first category* in the sense of Baire.

(ii) It is, of course, possible to provide conditions, in addition to those used in Theorem 3, that would guarantee that there are no purely patient points. For instance, it is easy to show, following the method of proof used by Diamond (1965, p. 175), that if a Paretian SWF, continuous in the sup metric, satisfies the non-complementarity axiom (introduced in Section 2.3), then the SWF must exhibit impatience at *every* non-constant utility stream. However, such results lead us away from the main line of inquiry of this paper.

(iii) The proof of Theorem 3 suggests that its conclusion holds for more general topological spaces. For any $m \in \mathbb{N}$, the *m*th *projection map* is a function $f_m : X \to Y$ satisfying $f_m(x) = x_m$. We let $\mathcal{T}$ denote the set of all topologies on *X* for which each projection map is continuous. If *W* is continuous on *X* with respect to a topology $\tau \in \mathcal{T}$, then *I* is an open subset of the topological space (X, τ). That is, continuity of *W* with respect to any topology in which the projection maps are continuous is sufficient to guarantee that the set of impatient points are open in that topology. (The sup metric topology is a special case of such a topology.)

(iv) If we are given a Paretian SWO $\succsim$ on *X*, which is continuous in the sup metric (in the sense that the upper and lower contour sets of each $x \in X$ is a closed set, in the sup metric), then $\succsim$ can be represented by a SWF, $W : X \to \mathbb{R}$, which is Paretian and which is continuous in the sup metric (see Diamond, 1965). Thus, Theorem 3 can be rewritten (in an equivalent form) in terms of a continuous Paretian SWO, and without any reference to a SWF. This reinterpretation of Theorem 3 is especially useful, since it tells us that while a Paretian SWO has *no* implications for impatience (as already noted above), a *continuous* (in the sup metric) Paretian SWO has the very strong implication that impatience is in fact *generic*.

[16] The set, *P*, of utility streams in *X*, at which *W* exhibits pure patience, is therefore a subset of a closed, nowhere dense set.

(v) Example 1 and Theorem 3 also lead us to make the following observation. For Paretian SWOs, the assumption of continuity (in the sup metric) of the SWO has significantly stronger implications for impatience than the assumption of its representability.

4. Proofs

We will first prove a technical result (Lemma 1) which generalizes the method of proof used by Basu and Mitra (2003) in establishing their impossibility result (Theorem 1 in their paper). The lemma will be used in establishing both the existence result (Theorem 1) and the denseness result (Theorem 2).

Lemma 1. *Let $W : X \to \mathbb{R}$ be any Paretian SWF. Let $y^0, y^1 \in Y$ with $y^0 < y^1$, $\bar{x} \in X$, and let $\mathbb{M} = \{t_1, t_2, t_3, \ldots\} \subset \mathbb{N}$ with $t_1 < t_2 < \ldots$. Define $X(\bar{x}; y^0, y^1, \mathbb{M}) = \{\tilde{x} \in X : \tilde{x}_t = \bar{x}_t$ if $t \in \mathbb{M}$ and $\tilde{x}_t \in \{y^0, y^1\}$ if $t \in \mathbb{M}\}$. Then, there exists some $\hat{x} \in X(\bar{x}; y^0, y^1, \mathbb{M})$ such that W exhibits impatience at $\hat{x}$.*

Proof. Let $W : X \to \mathbb{R}$ be a Paretian SWF and suppose there is some $\bar{x}$ and $\mathbb{M} = \{t_1, t_2, t_3, \ldots\} \subset \mathbb{N}$ with $t_1 < t_2 < \ldots$, such that for all $x \in X(\bar{x}; y^0, y^1, \mathbb{M})$, W exhibits pure patience at x.

To ease the writing, we may suppose that $y^1 = 1$ and $y^0 = 0$. Let Z denote the open interval (0, 1). Given the set $\mathbb{M}$, we can define a one-to-one correspondence, f, between the set $\mathbb{M}$ and the set of rationals in Z so that $f(t_k) = r_k$ for $k \in \mathbb{N}$.

Let z be an arbitrary number in Z. We define the set: $E(z) = \{t_n \in \mathbb{N} : r_n < z\}$. Then, $E(z)$ is an infinite set. We now define a sequence $a(z) = (a(z)_1, a(z)_2, \ldots)$ as follows:

$$a(z)_j = \begin{cases} 1, & \text{if } j = t_n \text{ for some } n \in \mathbb{N} \text{ and } t_n \in E(z) \\ 0, & \text{if } j = t_n \text{ for some } n \in \mathbb{N} \text{ and } t_n \notin E(z) \\ \bar{x}_j, & \text{if } j \notin \mathbb{M} \end{cases}$$

Note that the sequence will have an infinite number of 1's and an infinite number of 0's. Let us denote $\min\{t_j \in \mathbb{N} : a(z)_{t_j} = 0\}$ by $m(z)$. Since any non-empty subset of natural numbers has a minimum element, $m(z)$ is well-defined. Define the sequence $b(z) = (b(z)_1, b(z)_2, \ldots)$ as the same as the sequence $a(z)$ except that $a(z)_{m(z)} = 0$ and $b(z)_{m(z)} = 1$. By the Pareto axiom, we have $W(b(z)) > W(a(z))$. We denote the closed interval $[W(a(z)), W(b(z))]$ by $J(z)$. Note that for any $z \in Z$, we have $a(z), b(z) \in X(\bar{x}; 0, 1, \mathbb{M})$.

Let p, q be arbitrary real numbers in Z, with $q > p$. Clearly, we must have $E(p) \subset E(q)$, for if $t_n \in E(p)$, then $r_n < p$, and since $p < q$, we must have $r_n < q$, so that $t_n \in E(q)$. Further, there are an infinite number of rational numbers in the interval (p, q). Thus, comparing the sequence $a(p)$ with the sequence $a(q)$, we note that:

$$\text{(i) if } n \in \mathbb{N} \text{ and } a(p)_{t_n} = 1, \text{ then } a(q)_{t_n} = 1 \tag{2}$$

and

$$\text{(ii) there are an infinite number of } n \in \mathbb{N}, \text{ for which } a(p)_{t_n} = 0 \text{ and } a(q)_{t_n} = 1 \tag{3}$$

We can now follow the proof in Basu and Mitra (2003, pp. 1559–1560)[17] to show that:

$$W(a(q)) > W(b(p)) \tag{4}$$

[17] Note that the equality in display (4) in Basu and Mitra (2003, p. 1560) would be replaced by a weak inequality (following from the hypothesis that W exhibits pure patience at every $x \in X(\bar{x}; y^0, y^1, \mathbb{M})$), and this leads to the strict inequality (in display (4) here) just as it does in display (6) in Basu and Mitra (2003, p. 1560).

This means that the interval $J(q) = [W(a(q)), W(b(q))]$ is disjoint from the interval $[W(a(p)), W(b(p))]$, the latter interval lying entirely to the left of the former interval on the real line. That is, we have shown that the intervals $J(z)$ associated with distinct values of $z \in (0, 1)$ are non-overlapping. But, the set of all non-overlapping intervals in the reals is countable,[18] and this contradicts the uncountability of the reals in (0,1), establishing Lemma. □

Proof of Theorem 1. We know that there exist $y^0, y^1 \in Y$ with $y^0 < y^1$. To ease the writing, we may suppose that $y^1 = 1$ and $y^0 = 0$.

Let Z denote the open interval (0,1) and let $r_1, r_2, r_3, \ldots$ be an enumeration of the rational numbers in Z. Let z be an arbitrary number in Z. We define the set: $E(z) = \{n \in \mathbb{N} : r_n < z\}$. Clearly, $E(z)$ is an infinite set. We now define a sequence $a(z) = (a(z)_1, a(z)_2, \ldots)$ as follows:

$$a(z)_n = \begin{cases} 1, \text{ if } n \in E(z) \\ 0, \text{ otherwise} \end{cases}$$

Note that a is a map from Z to X. Denote the set of odd integers $\{1, 3, 5, \ldots\}$ by $\mathbb{M}$. For each $z \in Z$, define $x(z) \in X$ by: $x(z)_n = a(z)_{(n/2)}$ for $n \notin \mathbb{M}$ and $x(z)_n = 0$ for $n \in \mathbb{M}$.

Now, let p, q be arbitrary real numbers in Z, with $q > p$. There are an infinite number of rational numbers in the interval (p, q). Thus, comparing the sequence $a(p)$ with the sequence $a(q)$, we note that there are an infinite number of $n \in \mathbb{N}$, for which $a(p)_n = 0$ and $a(q)_n = 1$. This implies, in particular, that $a(p) \neq a(q)$ and so $x(p) \neq x(q)$. Thus, x is a one-to-one map from Z onto $x(Z)$.

Now, for each $z \in Z$, we define $X(x(z); 0, 1, \mathbb{M})$ as in Lemma 1. Then $I(x(z); 0, 1, \mathbb{M})$, the set of points in $X(x(z); 0, 1, \mathbb{M})$ which exhibit impatience, is non-empty for each $z \in Z$, by Lemma 1. Note that if $p, q \in Z$ and $p \neq q$, then $x(p) \neq x(q)$ and so $X(x(p); 0, 1, \mathbb{M}) \cap X(x(q); 0, 1, \mathbb{M}) = \emptyset$; this in turn implies that $I(x(p); 0, 1, \mathbb{M}) \cap I(x(q); 0, 1, \mathbb{M}) = \emptyset$. By the axiom of choice there is a function $g : Z \to X$, such that for each $z \in Z$, $g(z) \in I(x(z); 0, 1, \mathbb{M})$. Thus, g is a one-to-one correspondence from Z to $g(Z) \subset I$.

We have shown that I has a subset (namely $g(Z)$), which is equivalent to $Z = (0, 1)$ and so equivalent to $\mathbb{R}$, and so equivalent to $\mathbb{R}^{\mathbb{N}}$. (These latter two equivalences are well-known; see, for example, Kolmogorov and Fomin, 1970, p. 15 and p. 20.) On the other hand, $\mathbb{R}^{\mathbb{N}}$ has a subset, I, which is equivalent (trivially) to I. Thus, by the Cantor-Bernstein theorem,[19] I is equivalent to $\mathbb{R}^{\mathbb{N}}$. Hence, I has the power of the continuum, since $\mathbb{R}^{\mathbb{N}}$ has the power of the continuum. □

Proof of Theorem 2. Suppose on the contrary, there is some $x \in X$ and $\varepsilon > 0$ such that the open ball with center x and radius ε, denoted by $B(x, \varepsilon)$, is disjoint from I. Denote $\limsup x_n$ by c. There are two cases to consider: (i) $c = 0$ and (ii) $c > 0$.

In case (i), there is $N \in \mathbb{N}$, such that for all $n > N$, $x_n < (\varepsilon/2)$. Define $\mathbb{M} = \{N + 1, N + 2, \ldots\}$ and denoting $\min\{1, (\varepsilon/2)\}$ by δ, define $X(x; 0, \delta, \mathbb{M})$ as in Lemma 1, with $y^0 = 0$ and $y^1 = \delta$. By Lemma 1, there is $\hat{x} \in X(x; 0, \delta, \mathbb{M})$ such that $\hat{x} \in I$. Clearly, $d(x, \hat{x}) < \varepsilon$, and this contradiction establishes the result.

In case (ii), there is a subsequence $\{n_1, n_2, \ldots\}$ such that $|x_{n_k} - c| < (\varepsilon/2)$ for all $k \in \mathbb{N}$. Define $\mathbb{M} = \{n_1, n_2, \ldots\}$, and denoting $\min\{c, (\varepsilon/2)\}$ by δ, define $X(x; c - \delta, c, \mathbb{M})$ as in Lemma 1, with $y^0 = c - \delta$ and $y^1 = c$. By Lemma 1, there is $\hat{x} \in X(x; c - \delta, c, \mathbb{M})$ such that $\hat{x} \in I$. Clearly, $d(x, \hat{x}) < \varepsilon$, and this contradiction establishes the result.

[18] See, for example, Sierpinski (1965, p. 41).

[19] See, for example, Kolmogorov and Fomin (1970, p. 17).

Proof of Theorem 3. Let $\bar{x}$ be an arbitrary point in I. Then, there exist $M, N \in \mathbb{N}$, with $M > N$ such that $(\bar{x}_N - \bar{x}_M)(W(\bar{x}(M, N)) - W(\bar{x})) < 0$. Fix M, N in what follows. Observe that the function $g : X \to \mathbb{R}$ defined by $g(x) = W(x(M, N)) - W(x)$ is a continuous function on X. Since in the sup metric, every projection map is a continuous function, $h : X \to \mathbb{R}$ defined by $h(x) = x_N - x_M$ is continuous on X. This implies that the function $f : X \to \mathbb{R}$, defined by $f(x) = h(x)g(x)$, is a continuous function on X. By hypothesis, $f(\bar{x}) < 0$. By continuity of f at $\bar{x}$, there is an open set $O(\bar{x})$, containing $\bar{x}$, such that for every $x \in O(\bar{x})$, we have $f(x) < 0$. That is, $O(\bar{x}) \subset I$.

5. Concluding remarks

Theorem 2 provides a strong result on impatience, based solely on the existence of a Paretian SWF in the metric space (X, d). However, it is quite possible that stronger impatience implications might be obtained when a stronger notion of impatience is employed than the one used in this paper.

Example 1 indicates that, for Paretian SWFs, purely patient points can exist. But the SWF in this example is not continuous in the sup metric. We do not know of an example of a Paretian SWF, continuous in the sup metric, for which there is a purely patient point.

This leads us to end the paper on a somewhat speculative note. It might be the case that for Paretian SWFs, the assumption of continuity (in the sup metric) completely eliminates the possibility of pure patience. In this case, the impatience implication of continuity would be much stronger than that indicated in Theorems 2 and 3 of this paper.

On the other hand, it might be the case that the assumption of continuity (in the sup metric) does not eliminate pure patience, but severely limits the set of purely patient points in a way that allows us to say that this set is “negligible” from a measure-theoretic point of view, and therefore that the set of impatient points is *prevalent*.[20] Such a result would complement the observation, derived from Theorems 2 and 3, that impatience is generic (in the topological sense). We plan to explore these possibilities in future research.

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[20] In the language of prevalence analysis (see Anderson and Zame (2000), and the references cited there), one would have to show that the set of purely patient points is a subset of a *shy* set.

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